ON THE $q-$HERMITE POLYNOMIALS AND THEIR RELATIONSHIP WITH SOME OTHER FAMILIES OF ORTHOGONAL POLYNOMIALS

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ABSTRACT. We review properties of the $q-$Hermite polynomials and indicate their links with the Chebyshev, Rogers–Szegö, Al-Salam–Chihara, $q-$utraspherical polynomials. In particular we recall the connection coefficients between these families of polynomials. We also present some useful and important finite and infinite expansions involving polynomials of these families including symmetric and non-symmetric kernels. In the paper we collect scattered throughout literature useful but not widely known facts concerning these polynomials. It is based on 42 positions of predominantly recent literature.

1. Introduction

The aim of this paper is to review basic properties of the $q-$Hermite polynomials and collect their not always widely known properties scattered throughout recent literature. The $q-$Hermite polynomials constitute a $1-$parameter family of the orthogonal polynomials that for $q=1$ are well known Hermite polynomials, more precisely the probabilistic Hermite polynomials i.e. orthogonal with respect to the density of $N(0,1)$ distributions $(\exp(-x^2/2)/\sqrt{2\pi})$. For $q=0$ they are equal to the rescalled Chebyshev polynomials of the second kind, again more precisely, polynomials orthogonal with respect to the Wigner measure i.e. the one with the density $2\sqrt{4-x^2}/\pi$. On the other hand these polynomials are related to the so called Rogers-Szegö polynomials and other important families of polynomials such as the Al-Salam–Chihara polynomials.

Why these polynomials are important? For one thing they are very simple and as it will be shown in the sequel many more complicated (i.e. having more parameters) families of orthogonal polynomials can be expressed as linear combinations of $q-$Hermite polynomials. Secondly since they are simple many facts concerning them are known.

They appeared long time ago by the end of XIX-th century as a version of the Rogers polynomials (see [30], [29], [31]), their important properties were examined by Szegö [39] and Carlitz [10], [11], [9] through XX-th century, but only recently it appeared that they are important in noncommutative probability (see e.g. [3], [42]), quantum physics (see e.g. [14] [13]), combinatorics (see e.g. [19], [33], [18]) and last but not least ordinary, classical probability theory (see e.g. [4], [5], [8], [6], [24], [25]) extending the spectrum of known, finite support measures.

Date: December 8, 2010.

2010 *Mathematics Subject Classification.* Primary 33D45, 05A30; Secondary 42C05.

Key words and phrases. $q-$Hermite, Al-Salam-Chihara, Rogers-Szegö, Chebyshev polynomials, summing kernels, generalization of Poisson-Mehler kernel, generalization of Kibble-Slepian formula.

To define these polynomials and briefly describe their properties one has to adopt notation used in the so called q-series theory. Moreover the terminology concerning these polynomials is not fixed and under the same name appear sometimes different, but related to one another families of polynomials. Thus one has to be aware of these differences.

That is why the next section of the paper will be devoted to notation, definitions and discussion of different families of polynomials that function under the same name. The following section will be dedicated to the different 'finite expansions' formulae establishing relationships between these families of polynomials including listing known the so called 'connection coefficients' and 'linearization' formulae. The last section is dedicated to some infinite expansions involving discussed polynomials. It consists of three subsections first of which is devoted to different generalizations of the Mehler expansion formula, second to some useful, having auxiliary meaning, infinite expansions including reciprocals of some kernels. Finally the third one is dedicated to an attempt of a generalization of the 3- dimensional Kibble–Slepian formula with the Hermite polynomials substituted by the $q-$Hermite ones..

2. Notation and definitions of families of orthogonal polynomials

q is a parameter. We will assume that $-1 < q \leq 1$ unless otherwise stated. We will use traditional notation of the $q-$series theory i.e. $\left[0\right]_q = 0$; $\left[n\right]_q = 1 + q + \ldots + q^{n-1}$, $\left[n\right]_q! = \prod_{j=1}^{n}\left[j\right]_q$, with $\left[0\right]_q! = 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{\left[n\right]_q!}{\left[n-k\right]_q!\left[k\right]_q!} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases} .$$

It will be useful to use the so called $q-$Pochhammer symbol for $n \geq 1:$

$$\begin{aligned} \left(a;q\right)_n &= \prod_{j=0}^{n-1}\left(1 - aq^j\right), \\ \left(a_1, a_2, \ldots, a_k; q\right)_n &= \prod_{j=1}^{k}\left(a_j; q\right)_n . \end{aligned}$$

with $\left(a;q\right)_0 = 1$. Often $\left(a;q\right)_n$ as well as $\left(a_1, a_2, \ldots, a_k|q\right)_n$ will be abbreviated to $\left(a\right)_n$ and $\left(a_1, a_2, \ldots, a_k\right)_n$, if it will not cause misunderstanding.

It is easy to notice that $\left(q\right)_n = \left(1-q\right)^n\left[n\right]_q!$ and that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{\left(q\right)_n}{\left(q\right)_{n-k}\left(q\right)_k} & , \quad n \geq k \geq 0 \\ 0 & , \quad otherwise \end{cases} .$$

Notice that $\left[n\right]_1 = n, \left[n\right]_1! = n!$, $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$, $\left(a;1\right)_n = \left(1-a\right)^n$ and $\left[n\right]_0 = \begin{cases} 1 & if \quad n \geq 1 \\ 0 & if \quad n = 0 \end{cases}$, $\left[n\right]_0! = 1$, $\begin{bmatrix} n \\ k \end{bmatrix}_0 = 1$, $\left(a;0\right)_n = \begin{cases} 1 & if \quad n = 0 \\ 1-a & if \quad n \geq 1 \end{cases}$.

i will denote imaginary unit, unless otherwise clearly stated.

In the sequel we shall also use the following useful notation:

$$S\left(q\right) = \begin{cases} \left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right] & if \quad \left|q\right| < 1 \\ \mathbb{R} & if \quad q = 1 \end{cases} .$$

Sometimes we will define sets of polynomials only on bounded intervals. Of course they can be naturally extended to whole real line.

Basically each considered family of polynomials will be of one of two kinds. The first 'kind' will be orthogonal on $S(q)$ and their names will generally start with the capitals. The polynomials of the second 'kind' will be orthogonal on $[-1,1]$ and their names will generally will start with the lower case letters. There will be in fact 4 exceptions for the two kinds of Chebyshev polynomials (traditionally denoted by T and U), the so called Rogers or q-ultraspherical polynomials traditionally denoted by C and the Al-Salam–Chihara polynomials in its 'lower case version' version traditionally denoted by Q. These polynomials are orthogonal on $[-1,1]$ but their names as mentioned before traditionally start with the capital letter. The difference between those two kinds of polynomials are minor.

As of now it seems that the 'upper case' polynomials are more important in applications in the probability theory both commutative and noncommutative or quantum physics, while the 'lower case' polynomials are more typical in the special functions theory and the combinatorics.

In brief description of certain functions, given by infinite products, important for the discussed families of polynomials we will use the following families of auxiliary polynomials of degree at most 2.

In fact they are again of two different forms (as are families of polynomials) that are connected with the fact if considered polynomials are orthogonal on $[-1,1]$ regardless of q or on $S(q)$. As the families of polynomials these auxiliary polynomials will be denoted by the name starting with the capital if the case concerns orthogonality on $S(q)$.

Hence we will consider for $k\geq 0$:

$$\begin{aligned}
v_{k}\left(x,a|q\right) &= 1-2axq^{k}+a^{2}q^{2k}\\
V_{k}(x,a|q) &= 1-(1-q)axq^{k}+(1-q)a^{2}q^{2k}\\
w_{k}\left(x,y,t|q\right) &= \left(1-t^{2}q^{2k}\right)^{2}-4xytq^{k}(1+t^{2}q^{2k})+4t^{2}q^{2k}(x^{2}+y^{2})\\
W_{k}(x,y,t|q) &= \left(1-t^{2}q^{2k}\right)^{2}-(1-q)xytq^{k}(1+t^{2}q^{2k})+(1-q)t^{2}q^{2k}(x^{2}+y^{2})\\
l_{k}\left(x,a|q\right) &= \left(1+aq^{k}\right)^{2}-4x^{2}a^{2}q^{k}\\
L_{k}\left(x,a|q\right) &= \left(1+aq^{k}\right)^{2}-(1-q)x^{2}a^{2}q^{k}
\end{aligned}$$

Let us notice that

$$\begin{aligned}
v_{k}(x\sqrt{1-q}/2,a\sqrt{1-q}|q) &= V_{k}\left(x,a|q\right),\\
w_{k}\left(x\sqrt{1-q}/2,x\sqrt{1-q}/2,t|q\right) &= W_{k}\left(x,y,t|q\right),\\
l_{k}\left(x\sqrt{1-q}/2,a|q\right) &= L_{k}\left(x,a|q\right),\\
W_{k}\left(x,x,t|q\right) &= (1-tq^{k})^{2}L_{k}\left(x,t|q\right)
\end{aligned}$$

and that

$$\left(ae^{i\theta},ae^{-i\theta}\right)_{\infty} = \prod_{k=0}^{\infty}v_{k}\left(x,a|q\right), \tag{2.1}$$

$$\left(te^{i(\theta+\phi)},te^{i(\theta-\phi)},te^{-i(\theta-\phi)},te^{-i(\theta+\phi)}\right)_{\infty} = \prod_{k=0}^{\infty}w_{k}\left(x,y,t|q\right) \tag{2.2}$$

$$\left(ae^{2i\theta},ae^{-2i\theta}\right)_{\infty} = \prod_{k=0}^{\infty}l_{k}\left(x,a|q\right) \tag{2.3}$$

where, as usually in the $q-$series theory, $x=\cos\theta$ and $y=\cos\phi$.

The following convention will help in ordered listing of the properties of the discussed families of polynomials. Namely the family of polynomials whose names start with say a letter A will be referred to as A (similarly for lower case a) . There will be one exception namely members of the so called family of big q-Hermite polynomials are traditionally denoted by letter H (or h) as members of the family of $q-$Hermite polynomials. So family of big $q-$Hermite polynomials will be referred to by bH.

Let us also define the following sets of polynomials and present their generating functions and measures with respect to which these polynomials are orthogonal if these measures are positive.

2.1. **Hermite.** The Hermite polynomials are defined by the following $3-$term recurrence (2.4), below:

$$xH_{n}\left(x\right)=H_{n+1}\left(x\right)+nH_{n-1} \tag{2.4}$$

with $H_{0}\left(x\right)=H_{1}\left(x\right)=1$. They slightly differ from the Hermite polynomials h_{n} considered in most of the books on special functions. Namely

$$2xh_{n}\left(x\right)=h_{n+1}(x)+nh_{n-1}(x)$$

with $h_{-1}\left(x\right)=0$, $h_{0}\left(x\right)=1$.

It is known that polynomials $\{h_{n}\}$ are orthogonal with respect to $\exp\left(-x^{2}\right)$ while $\{H_{n}\}$ with respect to $\exp\left(-x^{2}/2\right)$. Moreover $H_{n}\left(x\right)=h_{n}\left(x/\sqrt{2}\right)/\left(\sqrt{2}\right)^{n}$. Besides we have

$$\exp\left(xt-t^{2}/2\right) = \sum_{k\geq0}\frac{t^{k}}{k!}H_{k}\left(x\right), \tag{2.5}$$

$$\exp\left(2xt-t^{2}\right) = \sum_{k\geq0}\frac{t^{k}}{k!}h_{k}\left(x\right). \tag{2.6}$$

2.2. **Chebyshev.** They are of two kinds. The Chebyshev polynomials of the first kind $\left\{T_{n}\right\}_{n\geq-1}$ are defined by the following $3-$term recursion

$$2xT_{n}\left(x\right)=T_{n+1}\left(x\right)+T_{n-1}\left(x\right), \tag{2.7}$$

for $n\geq1$, with $T_{0}\left(x\right)=1$, $T_{1}\left(x\right)=x$. One can define them also in the following way:

$$T_{n}\left(\cos\theta\right)=\cos\left(n\theta\right). \tag{2.8}$$

The Chebyshev polynomials $\{U_n(x)\}_{n\geq 0}$ of the second kind are defined by the same 3-term recurrence i.e. (2.7) with the different initial conditions, namely $U_0(x) = 1$ and $U_1(x) = 2x$. One shows that they can be defined also by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{2.9}$$

We have

$$\begin{aligned}\int_{-1}^{1} T_n(x)\,T_m(x)\,\frac{dx}{\pi\sqrt{1-x^2}} &= \left\{\begin{array}{ccc} 1 & if & m=n=0 \\ 1/2 & if & m=n\neq 0 \\ 0 & if & m\neq n\end{array}\right., \\ \int_{-1}^{1} U_n(x)\,U_m(x)\,\frac{2\sqrt{1-x^2}}{\pi}dx &= \left\{\begin{array}{ccc} 1 & if & m=n \\ 0 & if & m\neq n\end{array}\right..\end{aligned}$$

and for $|t|\leq 1$.

$$\sum_{k=0}^{\infty} t^k T_k(x) = \frac{1-tx}{1-2tx+t^2}, \tag{2.10}$$

$$\sum_{k=0}^{\infty} t^k U_k(x) = \frac{1}{1-2tx+t^2}. \tag{2.11}$$

2.3. **q-Hermite.** The $q-$Hermite polynomials are defined by:

$$h_{n+1}(x|q) = 2xh_n(x|q) - (1-q^n)h_{n-1}(x|q), \tag{2.12}$$

for $n\geq 1$ with $h_{-1}(x|q)=0$, $h_0(x|q)=1$. The Polynomials h_n are often called the continuous $q-$Hermite polynomials. In fact we will also use the following transformed form of the polynomials h_n, namely the polynomials:

$$H_n(x|q) = (1-q)^{-n/2} h_n\left(\frac{x\sqrt{1-q}}{2}|q\right). \tag{2.13}$$

We will call then also the $q-$Hermite polynomials. It is easy to notice that the polynomials $\{H_n(x|q)\}$ satisfy the following $3-$term recurrence

$$H_{n+1}(x|q) = xH_n(x|q) - [n]_q H_{n-1}(x), \tag{2.14}$$

for $n\geq 1$ with $H_{-1}(x|q)=0$, $H_1(x|q)=1$. The name is justified since one can easily show that $n\geq -1$

$$H_n(x|1) = H_n(x).$$

Notice that since $[n]_0 = 1$ for $n\geq -1$ we have

$$H_n(x|0) = U_n(x/2). \tag{2.15}$$

It is known that (see e.g. [17])

$$\begin{aligned}\int_{-1}^{1} h_n(x|q)\,h_m(x|q)\,f_N\left(\frac{2x}{\sqrt{1-q}}|q\right)\frac{2dx}{\sqrt{1-q}} &= \left\{\begin{array}{ccc} (q)_n & if & m=n \\ 0 & if & m\neq n\end{array}\right., \\ \int_{S(q)} H_n(x)\,H_m(x)\,f_N(x)\,dx &= \left\{\begin{array}{ccc} [n]_q! & if & m=n \\ 0 & if & m\neq n\end{array}\right.,\end{aligned}$$

where we denoted

$$f_N(x|q) = \left\{\begin{array}{ccc} \frac{(q)_\infty\sqrt{1-q}\prod_{j=0}^{\infty}L_j(x,1|q)}{2\pi\sqrt{L_0(x,1|q)}} & if & |q|<1 \\ \exp\left(-x^2/2\right)/\sqrt{2\pi} & if & q=1\end{array}\right., \tag{2.16}$$

and

$$\sum_{j=0}^{\infty}\frac{t^{j}}{(q)_{j}}h_{j}\left(x|q\right) = \frac{1}{\prod_{k=0}^{\infty}v_{k}\left(x,t|q\right)}, \tag{2.17}$$

$$\sum_{j=0}^{\infty}\frac{t^{j}}{[j]_{q}!}H_{j}\left(x|q\right) = \frac{1}{\prod_{k=0}^{\infty}V_{k}\left(x,t|q\right)}. \tag{2.18}$$

One proves also that

$$\begin{aligned}\lim_{q->1^{-}}f_{N}\left(x|q\right) &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^{2}}{2}\right),\\ \lim_{q->1^{-}}\frac{1}{\prod_{k=0}^{\infty}V_{j}\left(x,t|q\right)} &= \exp\left(xt-\frac{t^{2}}{2}\right).\end{aligned}$$

Rigorous and easy proof of this fact can be found in [19]. The convergence in distribution is obvious since we have $\forall n\geq 0:\lim_{q->1^{-}}H_{n}\left(x|q\right)=H_{n}\left(x|1\right)$ consequently we have the convergence of moments.

One considers also the small generalization of the $q-$Hermite polynomials namely the so called big continuous $q-$Hermite polynomials. i.e. the polynomials defined by the following 3-term recurrence:

$$\begin{aligned}(2x-aq^{n})h_{n}\left(x|a,q\right) &= h_{n+1}\left(x|a,q\right)+(1-q^{n})h_{n-1}\left(x|a,q\right)\\ (x-aq^{n})H_{n}\left(x|a,q\right) &= H_{n+1}\left(x|a,q\right)+[n]_{q}!H_{n-1}\left(x|a,q\right),\end{aligned}$$

with initial conditions: $h_{-1}\left(x|a,q\right)=H_{-1}\left(x\right)$ and $h_{0}\left(x|a,q\right)=H_{0}\left(x|a,q\right)=1$. They are obviously inter-related by

$$H_{n}\left(x|a,q\right)=h_{n}\left(\frac{\sqrt{1-q}x}{2}|a\sqrt{1-q},q\right)/\left(1-q\right)^{n/2}$$

Notice that, using well known properties of the ordinary Hermite polynomials, we have:

$$H_{n}\left(x|a,1\right)=H_{n}\left(x-a\right).$$

One can easily show (by calculating generating function and comparing it with (2.19), below and then applying (2.1)) that

$$h_{n}\left(x|a,q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}\left(ae^{i\theta}\right)_{k}e^{i(n-2k)\theta},$$

where as usually $x=\cos\theta$.

We have (see e.g. [23])

$$\sum_{j=0}^{\infty}\frac{t^{j}}{(q)_{j}}h_{j}\left(x|a,q\right) = \frac{(at)_{\infty}}{\prod_{k=0}^{\infty}v_{k}\left(x,t|q\right)}, \tag{2.19}$$

$$\sum_{j=0}^{\infty}\frac{t^{j}}{[j]_{q}!}H_{j}\left(x|a,q\right) = \frac{((1-q)at)_{\infty}}{\prod_{k=0}^{\infty}V_{k}\left(x,t|q\right)}. \tag{2.20}$$

We have also the following orthogonality relationships for $|a|<1$ (again e.g. [23])

$$\int_{S(q)}H_{n}\left(x|a,q\right)H_{m}\left(x|a,q\right)f_{bN}\left(x|a,q\right)=\left\{\begin{array}{ccc}0 & if & n\neq m\\ [n]_{q}! & if & n=m\end{array}\right.,$$

where

$$f_{bN}\left(x|a,q\right)=f_N\left(x|q\right)\frac{1}{\prod_{k=0}^{\infty}V_j\left(x,a|q\right)},$$

with similar formula for the polynomials $h_n\left(x|a,q\right)$. The family of the big $q-$Hermite polynomials will be referred to by symbol bH.

2.4. **Al-Salam–Chihara.** In the literature connected with the special functions as the Al-Salam–Chihara (ASC) function polynomials defined recursively:
(2.21)

$$Q_{n+1}\left(x|a,b,q\right)=(2x-(a+b)q^n)Q_n\left(x|a,b,q\right)-(1-abq^{n-1})(1-q^n)Q_{n-1}(x|a,b,q),$$

with $Q_{-1}\left(x|a,b,q\right)=0$, $Q_0\left(x|a,b,q\right)=1$. From Favard's theorem ([17]) it follows that if $|ab|\leq 1$, then there exists positive measure with respect to which polynomials Q_n are orthogonal. Further when $\left|a\right|,\left|b\right|<1$, then this measure has density. Since we are interested in the ASC polynomials in connection with the $q-$Hermite polynomials we will consider only the later case.

We will more often use these polynomials with new parameters ρ and y defined by $a=\frac{\sqrt{1-q}}{2}\rho(y-i\sqrt{\frac{4}{1-q}-y^2})$, $b=\frac{\sqrt{1-q}}{2}\rho(y+i\sqrt{\frac{4}{1-q}-y^2})$, such that $y^2\leq 4/(1-q)$, $|\rho|<1$. More precisely we will consider the polynomials
(2.22)

$$P_n\left(x|y,\rho,q\right)=Q_n\left(x\frac{\sqrt{1-q}}{2}|\frac{\rho\sqrt{1-q}}{2}(y-i\sqrt{\frac{4}{1-q}-y^2}),\frac{\rho\sqrt{1-q}}{2}(y+i\sqrt{\frac{4}{1-q}-y^2}),q\right)/(1-q)^{n/2}.$$

It is also of use to consider another version of the ASC polynomials, namely for $\left|x\right|,\left|y\right|,\left|\rho\right|,\left|q\right|<1:$

$$p_n\left(x|y,\rho,q\right)=P_n\left(\frac{2x}{\sqrt{1-q}}|\frac{2y}{\sqrt{1-q}},\rho,q\right). \tag{2.23}$$

One can easily show that the polynomials P_n and p_n satisfy the following $3-$term recurrence:

$$P_{n+1}(x|y,\rho,q)= \tag{2.24}$$

$$(x-\rho yq^n)P_n(x|y,\rho,q)-(1-\rho^2q^{n-1})[n]_qP_{n-1}(x|y,\rho,q), \tag{2.25}$$

$$p_{n+1}\left(x|y,\rho,q\right)= \tag{2.26}$$

$$2(x-\rho yq^n)p_n\left(x|y,\rho,q\right)-\left(1-\rho^2q^{n-1}\right)(1-q^n)p_n\left(x|y,\rho,q\right) \tag{2.27}$$

with $P_{-1}\left(x|y,\rho,q\right)=p_{-1}\left(x|y,\rho,q\right)=0$, $P_0\left(x|y,\rho,q\right)=p_0\left(x|y,\rho,q\right)=1$ since obviously $a+b=\rho y\sqrt{1-q}$ and $ab=\rho^2$ in the case of the polynomials P and $a+b=2\rho y$ and $ab=\rho^2$ in the case of the polynomials p.

The polynomials $\{P_n\}$ have a nice probabilistic interpretation see e.g. [5]. To support intuition let us notice that

$$\begin{aligned}
P_n\left(x|y,\rho,1\right) &= (1-\rho^2)^{n/2}H_n\left(\frac{x-\rho y}{\sqrt{1-\rho^2}}\right),\\
P_n\left(x|y,\rho,0\right) &= U_n\left(x/2\right)-\rho yU_{n-1}\left(x/2\right)+\rho^2U_{n-2}\left(x/2\right),
\end{aligned}$$

if we define $U_{-r}\left(x\right)=0$, $r\geq 1$.

We have the following orthogonality relationships (see [17]):

$$\int_{-1}^{1} Q_n\left(x|a,b,q\right) Q_m\left(x|a,b,q\right)\omega\left(x|a,b,q\right)dx = \begin{cases} 0 & if \quad n \neq m \\ \left(q\right)_n \left(ab\right)_n & if \quad m = m \end{cases}$$

where

$$\omega\left(x|a,b,q\right) = \frac{\left(q\right)_\infty \left(ab\right)_\infty}{2\pi\sqrt{1-x^2}} \prod_{k=0}^{\infty} \frac{l_k\left(x,1|q\right)}{((1-abq^{2k})^2 - 2x(a+b)q^k(1+abq^{2k}) + q^{2k}ab(4x^2+(a+b)/(ab))}.$$

Also after passing to the parameters ρ and y we get (see [5]):
(2.28)

$$\int_{S(q)} P_n(x|y,\rho,q)P_m\left(x|y,\rho,q\right)f_{CN}\left(x|y,\rho,q\right)dx = \begin{cases} 0 & if \quad m \neq n \\ \left[n\right]_q! \left(\rho^2\right)_n & if \quad m = n \end{cases},$$

where we denoted for $|q| < 1$:

$$f_{CN}\left(x|y,\rho,q\right) = \frac{\sqrt{1-q}\left(q\right)_\infty \left(\rho^2\right)_\infty}{2\pi\sqrt{L_0\left(x,1|q\right)}} \prod_{k=0}^{\infty} \frac{L_k\left(x,1|q\right)}{W_k\left(x,y,\rho|q\right)}. \tag{2.29a}$$

Let us notice that :

$$f_{CN}\left(x|y,\rho,q\right) = f_N\left(x|q\right)\frac{\left(\rho^2\right)_\infty}{\prod_{k=0}^{\infty} W_k\left(x,y,\rho|q\right)}.$$

We also set

$$f_{CN}\left(x|y,\rho,1\right) = \frac{1}{\sqrt{2\pi\left(1-\rho^2\right)}}\exp\left(-\frac{\left(x-\rho y\right)^2}{2\left(1-\rho^2\right)}\right).$$

Again one shows (see e.g. [19]) that

$$f_{CN}\left(x|y,\rho,q\right) \underset{q\to 1^-}{\longrightarrow} f_{CN}\left(x|y,\rho,1\right).$$

One shows (see e.g. [5]) that for $|x|$,$|z| \in S\left(q\right)$:

$$\int_{S(q)} f_{CN}\left(x|y,\rho_1,q\right) f_{CN}\left(y|,z,\rho_2,q\right)dy = f_{CN}\left(x|z,\rho_1\rho_2,q\right).$$

We also have

$$\sum_{k=0}^{\infty} \frac{t^k}{\left(q\right)_k} Q_k\left(x|a,b,q\right) = \frac{\left(at,bt\right)_\infty}{\prod_{j=0}^{\infty} v_j\left(x,t|q\right)},$$

and for the parameters ρ and y :

$$\sum_{k=0}^{\infty} \frac{t^k}{\left[k\right]_q!} P_k\left(x|y,\rho,q\right) = \prod_{j=0}^{\infty} \frac{V_j\left(y,\rho t|q\right)}{V_j\left(x,t|q\right)}.$$

2.5. $q-$**utraspherical polynomials.** It turns out that the polynomials $\left\{H_n\right\}_{\geq -1}$ are also related to another family of orthogonal polynomials $\left\{C_n\left(x|\beta,q\right)\right\}_{n\geq -1}$ which was considered by Rogers in 1894 (see [29]). Now they are called the $q-$utraspherical polynomials. The polynomials C_n can be defined through their $3-$recurrence:

$$(1-q^{n+1})C_{n+1}\left(x|\beta,q\right) = 2(1-\beta q^n)xC_n(x|\beta,q) - (1-\beta^2 q^{n-1})C_{n-1}\left(x|\beta,q\right),$$

for $n\geq 0$, with $C_{-1}\left(x|\beta,q\right)=0$, $C_{0}\left(x|\beta,q\right)=1$, where β is a real parameter such that $|\beta|<1$. One shows (see e.g. [17]) that

$$C_{n}\left(x|\beta,q\right)=\sum_{k=0}^{n}\frac{\left(\beta\right)_{k}\left(\beta\right)_{n-k}}{\left(q\right)_{k}\left(q\right)_{n-k}}e^{i(n-2k)\theta},$$

where $x=\cos\theta$. Hence we have (following formula (2.35):

$$C_{n}\left(x|0,q\right)=\frac{h_{n}\left(x|q\right)}{\left(q\right)_{n}}.$$

In fact we will consider slightly modified polynomials C_{n}. Namely we will consider polynomials $R_{n}\left(x|\beta,q\right)$ related to polynomials C_{n} through the relationship:

$$C_{n}\left(x|\beta,q\right)=\left(1-q\right)^{n/2}R_{n}\left(\frac{2x}{\sqrt{1-q}}|\beta,q\right)/\left(q\right)_{n},n\geq 1. \tag{2.30}$$

One can easily check that the polynomials $\{R_{n}\}$ satisfy the following 3−term recurrence:

$$R_{n+1}\left(x|\beta,q\right)=\left(1-\beta q^{n}\right)xR_{n}\left(x|\beta,q\right)-\left(1-\beta^{2}q^{n-1}\right)\left[n\right]_{q}R_{n-1}\left(x|\beta,q\right). \tag{2.31}$$

We have an easy Proposition

Proposition 1. *For $n\geq 1$: i) $R_{n}\left(x|0,q\right)=H_{n}\left(x|q\right)$,*
ii) $R_{n}\left(x|q,q\right)=\left(q\right)_{n}U_{n}\left(x\sqrt{1-q}/2\right)$,
iii) $\lim_{\beta->1^{-}}\frac{R_{n}\left(x|\beta,q\right)}{\left(\beta\right)_{n}}=2\frac{T_{n}\left(x\sqrt{1-q}/2\right)}{\left(1-q\right)^{n/2}}$

Proof. i) direct calculation. ii) We have for $\beta=q$: $\tilde{R}_{n+1}\left(x|q,q\right)=x\tilde{R}_{n}\left(x|q,q\right)-\tilde{R}_{n-1}\left(x|q,q\right)$, where we denoted $\tilde{R}_{n}(x|q,q)=(1-q)^{n/2}R_{n}\left(x|q,q\right)/\left(q\right)_{n}$. So visibly since $R_{0}\left(x|q,q\right)=1$ and $R_{1}\left(x|q,q\right)=x$. iii) Let us first denote $F_{n}\left(x|\beta,q\right)=\frac{R_{n}\left(x|\beta,q\right)}{\left(\beta\right)_{n}}$, write the 3−term recurrence for it obtaining $F_{n+1}\left(x|\beta,q\right)=xF_{n}\left(x|\beta,q\right)$ $-\frac{(1-q^{n})(1-\beta^{2}q^{n-1})}{(1-q)(1-\beta q^{n})(1-\beta q^{n-1})}F_{n-1}\left(x|\beta,q\right)$, with $F_{-1}\left(x|\beta,q\right)=0$, $F_{0}\left(x|\beta,q\right)=1$ and let $\beta->1^{-}$. We immediately see that the limit, denote it by $F_{n}\left(x|1,q\right)$ satisfies the following the 3−term recurrence: $F_{n+1}\left(x|1,q\right)=xF_{n}\left(x|1,q\right)-\frac{F_{n-1}\left(x|1,q\right)}{\left(1-q\right)}$, which confronted with the 3−term recurrence satisfied by the polynomials T_{n} proves our assertion. □

It is known that (see e.g. [17]):

$$\int_{-1}^{1}C_{n}\left(x|\beta,q\right)C_{m}\left(x|\beta,q\right)f_{R}\left(\frac{2x}{\sqrt{1-q}}|\beta,q\right)\frac{2}{\sqrt{1-q}}dx=\begin{cases}0 & if\quad m\neq n\\ \frac{(1-\beta)\left(\beta^{2}\right)_{n}}{(1-\beta q^{n})\left(q\right)_{n}} & if\quad m=n\end{cases}$$

$$\int_{S(q)}R_{n}\left(x|\beta,q\right)R_{m}\left(x|\beta,q\right)f_{R}\left(x|\beta,q\right)=\begin{cases}0 & when\quad n\neq m\\ \frac{(1-\beta)\left(\beta^{2}\right)_{n}[n]_{q}!}{(1-\beta q^{n})} & when\quad n=m\end{cases}$$

where we denoted

$$f_{R}\left(x|\beta,q\right)=\frac{\left(q,\beta^{2}\right)_{\infty}\sqrt{1-q}}{\left(\beta,\beta q\right)_{\infty}2\pi\sqrt{L_{0}\left(x,1|q\right)}}\prod_{k=0}^{\infty}\frac{L_{k}\left(x,1|q\right)}{L_{k}\left(x,\beta|q\right)}. \tag{2.32}$$

Let us remark that

$$f_{R}\left(x|\beta,q\right)=f_{N}\left(x|q\right)\frac{\left(\beta^{2}\right)_{\infty}}{\left(\beta,\beta q\right)_{\infty}\prod_{k=0}^{\infty}L_{k}\left(x,\beta|q\right)}.$$

Notice also that examining the 3-term recurrence satisfied by P_n and $R_n$we see $\forall n\geq -1:$

$$P_{n}\left(x|x,\rho,q\right)=R_{n}\left(x|\rho,q\right),$$

and that for $\left\vert x\right\vert ,\left\vert y\right\vert \in S\left(q\right)$

$$f_{CN}\left(x|x,\rho,q\right)/(1-\rho)=f_{R}\left(x|\rho,q\right),$$

since we have $(1-\rho^2q^{2k})^2-(1-q)\rho q^k(1+\rho^2q^{2k})x^2+2(1-q)\rho^2x^2q^{2k}=\left(1-\rho q^k\right)^2\left(\left(1+\rho q^k\right)^2\right.$ $\left.-(1-q)\rho x^2q^k\right)$ and the fact that $\frac{(\rho)_\infty(\rho q)_\infty}{(\rho)_\infty^2}=\frac{1}{1-\rho}$.

We also have

$$\sum_{k=0}^{\infty}t^{k}C_{k}\left(x|\beta,q\right) = \prod_{k=0}^{\infty}\frac{v_{k}\left(x,\beta t|q\right)}{v_{k}\left(x,t|q\right)}, \tag{2.33}$$

$$\sum_{k=0}^{\infty}\frac{t^{k}}{\left[k\right]_{q}!}R_{k}\left(x|\beta,q\right) = \prod_{j=0}^{\infty}\frac{V_{k}\left(x,\beta t|q\right)}{V_{k}\left(x,t|q\right)}. \tag{2.34}$$

Remark 1. Assertion ii) of Proposition 1 could have been deduced also from (2.34), namely putting $\beta=q$ we get $\sum_{k=0}^{\infty}\frac{t^{k}}{\left[k\right]_{q}!}R_{k}\left(x|q,q\right)=\frac{1}{(1-(1-q)tx+(1-q)t^{2})}$which confronted with (2.11) and formula $\left(q\right)_{n}=\left(1-q\right)^{n}\left[n\right]_{q}!$ leads to the conclusion that $R_{k}\left(x|q,q\right)/\left(q\right)_{k}=U_{k}\left(x\sqrt{1-q}/2\right)$. Following this idea we see that $\sum_{k=0}^{\infty}\frac{t^{k}}{\left[k\right]_{q}!}R_{k}\left(x|q^{2},q\right)$ $=\frac{1}{(1-(1-q)tx+(1-q)t^{2})(1-(1-q)txq+(1-q)t^{2}q^{2})}$.
Hence $R_{k}\left(x|q^{2},q\right)/\left(q\right)_{k}=\sum_{k=0}^{n}q^{k}U_{k}\left(x\sqrt{1-q}/2\right)U_{n-k}\left(x\sqrt{1-q}/2\right)$ using common knowledge on the properties of the generating functions. Simple 'generating functions' argument shows that $\sum_{k=0}^{n}q^{k}U_{k}\left(x\sqrt{1-q}/2\right)U_{n-k}\left(x\sqrt{1-q}/2\right)$ simplifies to $\sum_{j=0}^{\left\lfloor n/2\right\rfloor}q^{j}\left[n+1-2j\right]_{q}U_{n-2j}\left(x\sqrt{1-q}/2\right)$. On the other hand since these polynomials are proportional to $R_{k}\left(x|q^{2},q\right)$ we know their 3-term recurrence and the density that makes them orthogonal. Similarly for other cases $R_{k}\left(x|q^{m},q\right)$, $m\geq3$. Besides notice that $\forall n\geq-1,x\in\mathbb{R}\ \lim_{m\rightarrow\infty}R_{k}\left(x|q^{m},q\right)=H_{n}\left(x|q\right)$.

We will need also two families of auxiliary polynomials.

2.6. **Rogers-Szegö.** These polynomials are defined by the equality:

$$s_{n}\left(x|q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}x^{k},$$

for $n\geq0$ and $W_{-1}\left(x|q\right)=0$. They will be playing here an auxiliary rôle. In particular one shows (see e.g. [17]) that:

$$h_{n}\left(x|q\right)=e^{in\theta}s_{n}\left(e^{-2i\theta}|q\right) \tag{2.35}$$

where $x=\cos\theta$, and that:

$$\sup_{\left\vert x\right\vert\leq1}\left\vert h_{n}\left(x|q\right)\right\vert\leq s_{n}\left(1|q\right).$$

In the sequel the following identities discovered by Carlitz (see Exercise 12.2(b) and 12.2(c) of [17]), true for $\left\vert q\right\vert ,\left\vert t\right\vert <1$:

$$\sum_{k=0}^{\infty}\frac{s_{k}\left(1|q\right)t^{k}}{\left(q\right)_{k}}=\frac{1}{\left(t\right)_{\infty}^{2}},\sum_{k=0}^{\infty}\frac{s_{k}^{2}\left(1|q\right)t^{k}}{\left(q\right)_{k}}=\frac{\left(t^{2}\right)_{\infty}}{\left(t\right)_{\infty}^{4}}, \tag{2.36}$$

will allow to show convergence of many considered in the sequel series.

2.7. $q^{-1}-$**Hermite.** We will need also polynomials $\{B_n(x|q)\}_{n\geq -1}$ defined by the following 3-term recurrence:

$$B_{n+1}(y|q) = -q^n y B_n(y|q) + q^{n-1}[n]_q B_{n-1}(y|q)\,; n\geq 0 \tag{2.37}$$

with $B_{-1}(y|q) = 0$, $B_0(y|q) = 1$. One easily shows that $B_n(x|1) = i^n H_n(ix)$ (compare [5]). We will also sometimes need the 'continuous version' of these polynomials namely $b_n(y|q) = (1-q)^{n/2} B_n\left(2y/\sqrt{1-q}|q\right)$. It is easy to notice that the polynomials b_n satisfy the following $3-$term recurrence :

$$b_{n+1}(y|q) = -2q^n y b_n(y|q) + q^{n-1}(1-q^n) b_{n-1}(y|q)\,, \tag{2.38}$$

with $b_{-1}(y|q) = 0, b_0(y|q) = 1$. Moreover if we consider $\tilde{b}_n(y|q) = q^{-n(n-1)/2} i^n b_n(iy|q)$ then we see that the polynomials $\tilde{b}_n$ satisfy the following 3-term recurrence:

$$\tilde{b}_{n+1}(y|q) = 2y\tilde{b}_n(y|q) - (q^{-n}-1)\tilde{b}_{n-1}(y|q)\,,$$

hence $\tilde{b}_n's$ are orthogonal for $q > 1$. The point is that there does not exit the unique measure that makes these polynomials orthogonal. Discussion of this case is thoroughly done in [21]. However polynomials $\{B_n\}$ will be of great help in the sequel.

Notice ([5]) that

$$\sum_{k=0}^{\infty} \frac{t^k}{[k]_q!} B_k(x|q) = \prod_{j=0}^{\infty} V_j(x,t|q)\,.$$

Hence in particular we get: $B_n(y|0) = \begin{cases} -y & if \quad n=1 \\ 1 & if \quad n=2\vee n=0 \\ 0 & if \quad n\geq 3 \end{cases}$.

3. Connection coefficients and other useful finite expansions

3.1. **Connection coefficients.** We consider $n\geq 0$

T&U

$$\begin{aligned} T_n(x) &= (U_n(x) - U_{n-2}(x))/2, \\ U_n(x) &= 2\sum_{k=0}^{\lfloor n/2\rfloor} T_{n-2k}(x) - (1+(-1)^n)/2. \end{aligned}$$

These expansions belong to common knowledge of the special functions theory

H&T

$$H_n(x|q) = (1-q)^{-n/2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q T_{n-2k}\left(x\sqrt{1-q}/2\right),$$

if one sets $T_{-n}(x) = T_n(x)$, $n\geq 0$.

First notice that (2.35) is equivalent to $h_n(x) = \sum_{k=0}^{n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q \cos(2k-n)\theta$ where $x = \cos\theta$. Next we use (2.8)

H&H

$$H_n(x|p) = \sum_{k=0}^{\lfloor n/2\rfloor} C_{n,n-2k}(p,q) H_{n-2k}(x|q)\,,$$

where

$$C_{n,n-2k}(p,q)=\frac{(1-q)^{n/2-k}}{(1-p)^{n/2}}\times$$

$$\sum_{j=0}^{k}(-1)^{j}p^{k-j}q^{j(j+1)/2}\begin{bmatrix}n-2k+j\\ j\end{bmatrix}_{q}(\begin{bmatrix}n\\ k-j\end{bmatrix}_{p}-p^{n-2k+2j+1}\begin{bmatrix}n\\ k-j-1\end{bmatrix}_{p}).$$

This formula follows the 'change of base' formula for the continuous $q-$Hermite polynomials (i.e. polynomials h_n) in e.g. [18], [2] or [15] (formula 7.2) that states that:

$$h_{n}\left(x|p\right)=\sum_{k=0}^{\lfloor n/2\rfloor}c_{n,n-2k}\left(p,q\right)h_{n-2k}\left(x|q\right)$$

where

$$c_{n,n-2k}\left(p,q\right)=\frac{\left(1-p\right)^{n/2}}{\left(1-q\right)^{n/2-k}}C_{n,n-2k}(p,q).$$

U&H

$$U_{n}\left(x\sqrt{1-q}/2\right)=\sum_{j=0}^{\lfloor n/2\rfloor}\left(-1\right)^{j}\left(1-q\right)^{n/2-j}q^{j(j+1)/2}\begin{bmatrix}n-j\\ j\end{bmatrix}_{q}H_{n-2j}\left(x|q\right),$$

$$H_{n}\left(y|q\right)=\sum_{k=0}^{\lfloor n/2\rfloor}\left(1-q\right)^{-n/2}\frac{q^{k}-q^{n-k+1}}{1-q^{n-k+1}}\begin{bmatrix}n\\ k\end{bmatrix}_{q}U_{n-2k}\left(y\sqrt{1-q}/2\right).$$

These expansion follow the previous one setting once $p=0$ and then secondly $q=0$ and then $p=q$.

H&bH

$$h_{n}\left(x|a,q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_{q}(-1)^{k}q^{\binom{k}{2}}a^{k}h_{n-k}\left(x|q\right),\tag{3.1}$$

$$H_{n}\left(x|a,q\right)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_{q}(-1)^{k}q^{\binom{k}{2}}a^{k}H_{n-k}\left(x|q\right).\tag{3.2}$$

(3.1) it is formula 19 of [12] (see also [13]). (3.2) is a simple consequence of (3.1).

H&P

$$P_{n}\left(x|y,\rho,q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}\rho^{n-j}B_{n-j}\left(y|q\right)H_{j}\left(x|q\right),\tag{3.3}$$

$$H_{n}\left(x|q\right)=\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}\rho^{n-j}H_{n-j}\left(y|q\right)P_{j}\left(x|y,\rho,q\right).\tag{3.4}$$

For the proof of (3.3) see Remark 1 following Theorem 1 in [5]. For the proof of (3.4) we start with formula (4.7) in [20] that gives connection coefficients of h_n with respect to Q_n. Then we pass to the polynomials H_n & P_n using formulae $h_n\left(x|q\right)=\left(1-q\right)^{n/2}H_n\left(\frac{2x}{\sqrt{1-q}}|q\right)$, $n\geq 1$ and $p_n(x|a,b,q)=\left(1-q\right)^{n/2}P_n\left(\frac{2x}{\sqrt{1-q}}|\frac{2a}{\sqrt{(1-q)b}},\sqrt{b},q\right)$. By the way notice that this formula can be easily derived from assertions iv) and

(3.20) with $m=0$ presented below and the standard change of order of summation. Now it remains to return to polynomials H_n .

As a corollary of (3.4) and (2.28) we get a nice formula given in [5]: For $\forall n\geq 1$, $|\rho|<1$,$y\in S\left(q\right)$

$$\int_{S(q)}H_{n}\left(x|q\right)f_{CN}\left(x|y,\rho,q\right)dx=\rho^{n}H_{n}\left(y|q\right)$$

bH&P

$$H_{n}\left(x|a,q\right) = \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{q}P_{j}\left(x|y,\frac{a}{b},q\right)\left(\frac{a}{b}\right)^{n-j}H_{n-j}\left(y|b,q\right), \tag{3.5}$$

$$P_{n}\left(x|y,\rho,q\right) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q}\rho^{n-k}B_{n-k}\left(x|a/\rho,q\right)H_{k}\left(x|a,q\right), \tag{3.6}$$

where we denoted $B_{m}\left(x|b,q\right)\overset{df}{=}\sum_{j=0}^{m}\begin{bmatrix}m\\j\end{bmatrix}_{q}b^{m-j}B_{j}\left(x|q\right)$. Strict proof of (3.5) and (3.6) is presented in [37]. It is easy and is based on (3.3) and (3.2).

P&P

$$P_{n}\left(x|y,\rho,q\right) = \sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_{q}r^{n-j}P_{j}\left(x|z,r,q\right)P_{n-j}(z|y,\rho/r,q), \tag{3.7}$$

$$\frac{P_{n}\left(y|z,t,q\right)}{(t^{2})_{n}} = \sum_{j=0}^{n}(-1)^{j}q^{j(j-1)/2}\begin{bmatrix}n\\j\end{bmatrix}_{q}t^{j}H_{n-j}\left(y|q\right)\frac{P_{j}\left(z|y,t,q\right)}{(t^{2})_{j}}, \tag{3.8}$$

if one extends definition of polynomials P_n for $|\rho|>1$ by (3.3). (3.7) has been proved in [37], while (3.8) is given in [36] Corollary 2. Besides it follows directly from one of the infinite expansions that will be presented in section 4.

As a corollary of (3.8) and of course (2.28) we get the following formula: For $\forall n\geq 1,|\rho|<1$,$x\in S\left(q\right)$

$$\int_{S(q)}P_{n}\left(x|y,\rho,q\right)f_{CN}\left(y|x,\rho,q\right)dy=\left(\rho^{2}\right)_{n}H_{n}\left(x|q\right).$$

R&R

For $|\beta|,|\gamma|<1$:

$$R_{n}\left(x|\gamma,q\right)=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{[n]_{q}!\beta^{k}\left(\gamma/\beta\right)_{k}\left(\gamma\right)_{n-k}\left(1-\beta q^{n-2k}\right)}{[k]_{q}!\left[n-2k\right]_{q}!\left(\beta q\right)_{n-k}\left(1-\beta\right)}R_{n-2k}\left(x|\beta,q\right). \tag{3.9}$$

(3.9) is in fact celebrated connection coefficient formula for the Rogers polynomials which was in fact expressed in terms of polynomials C_n.

R&H

For $|\beta|,|\gamma|<1$:

$$R_{n}\left(x|\gamma,q\right) = \sum_{k=0}^{\lfloor n/2\rfloor}(-1)^{k}\frac{q^{k(k-1)/2}\left[n\right]_{q}!\gamma^{k}\left(\gamma\right)_{n-k}}{[k]_{q}!\left[n-2k\right]_{q}!}H_{n-2k}\left(x|q\right), \tag{3.10}$$

$$H_{n}\left(x|q\right) = \sum_{k=0}^{\lfloor n/2\rfloor}\frac{[n]_{q}!}{[k]_{q}!\left[n-2k\right]_{q}!}\frac{\beta^{k}\left(1-\beta q^{n-2k}\right)}{\left(1-\beta\right)\left(\beta q\right)_{n-k}}R_{n-2k}\left(x|\beta,q\right). \tag{3.11}$$

(3.10) and (3.11) are particular cases of (3.9), first for $\beta = 0$ and the second for $\gamma = 0$.

B&H

$$B_n(x|q) = (-1)^n q^{\binom{n}{2}} \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ k \end{bmatrix}_q [k]_q! q^{k(k-n)} H_{n-2k}(x|q). \tag{3.12}$$

(3.12) was proved in [36] Lemma 2 assertion i).

As an immediate observation we have the following expansion of the ASC polynomials in the q-Hermite polynomials.

Proposition 2.

$$P_n(x|y,\rho,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ k \end{bmatrix}_q [k]_q! q^{k(k-1)} \rho^{2k} \times$$
$$\sum_{s=0}^{n-2k} (-1)^s \begin{bmatrix} n-2k \\ s \end{bmatrix}_q q^{\binom{s}{2}} (q^k \rho)^s H_{n-2k-s}(x|q) H_s(y|q)$$

Proof. First we use (3.3) and then (3.12) obtaining: $P_n(x|y,\rho,q) = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q H_{n-s}(x|q) \rho^s (-1)^s q^{\binom{s}{2}}$ $\times \sum_{k=0}^{\lfloor s/2 \rfloor} \begin{bmatrix} s \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ k \end{bmatrix}_q [k]_q! q^{k(k-s)} H_{s-2k}(y|q)$. Now we change the order of summation. □

3.2. **Useful finite expansions.** We start with the so called 'linearization formulae'. These are the formulae expressing a product of two or more polynomials of the same type as linear combinations of polynomials of the same type as the ones produced. We will extend the name 'linearization formulae' by relaxing the requirement of polynomials involved to be of the same type. Generally to obtain 'linearization formula ' is not simple and requires a lot of tedious calculations.

3.2.1. *Linearization formulae.* $q-$Hermite polynomials

The formulae below can be found in e.g. [17] (Thm. 13.1.5) and also in [1] and originally were formulated for polynomials h_n. Here below are presented for polynomials H_n using (2.13):

$$H_n(x|q) H_m(x|q) = \sum_{j=0}^{\min(n,m)} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q [j]_q! H_{n+m-2k}(x|q), \tag{3.13}$$

$$H_n(x|q) H_m(x|q) H_k(x|q) = \tag{3.14}$$

$$\sum_{r,s} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} k \\ s \end{bmatrix}_q \begin{bmatrix} m+n-2r \\ s \end{bmatrix}_q [s]_q! [r]_q! H_{n+m+k-2r-2s}(x|q) = \tag{3.15}$$

(3.16)
$$\sum_{j=0}^{\lfloor (k+m+n)/2 \rfloor} \left(\sum_{r=\max(j-k,0)}^{\min(m,n,m+n-j)} \begin{bmatrix} m \\ r \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} k \\ j-r \end{bmatrix}_q \begin{bmatrix} m+n-2r \\ j-r \end{bmatrix}_q [r]_q [j-r]_q \right) H_{n+m+k-2j}(x|q).$$

In fact (3.13) can be easily derived (by re-scaling and changing of variables) from an old result of Carlitz ([11]) that was formulated in terms of the Rogers-Szegö $\{s_n(x|q)\}_{n\geq -1}$ polynomials. Carlitz proved in the same paper another useful

identity concerning polynomials s_n that can be easily reformulated in terms of the polynomials H_n. The formula below is in a sense an inverse of (3.13). Namely we have:

$$H_{n+m}(x)=\sum_{k=0}^{\min(n,m)}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}m\\k\end{bmatrix}_q\begin{bmatrix}n\\k\end{bmatrix}_q[k]_q!H_{n-k}(x)H_{m-k}(x).\tag{3.17}$$

H&B

$\forall n,m\geq 1:$

(3.18)

$$H_m(x|q)B_n(x|q)=(-1)^n q^{\binom{n}{2}}\sum_{k=0}^{\lfloor(n+m)/2\rfloor}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n+m-k\\k\end{bmatrix}_q[k]_q!q^{-k(n-k)}H_{n+m-2k}(x|q).$$

This formula having technical importance has been proved in [36] Lemma 2 assertion ii).

H&R

We have also useful formula:

$\forall n,m\geq 1:$

(3.19)

$$H_m(x|q)R_n(x|\beta,q)=\sum_{k,j}\begin{bmatrix}m\\j\end{bmatrix}_q\begin{bmatrix}n\\k+j\end{bmatrix}_q\begin{bmatrix}n-k-j\\k\end{bmatrix}[k+j]_q!(-\beta)^k q^{\binom{k}{2}}(\beta)_{n-k}H_{n+m-2k-2j}(x|q),$$

Which was proved in [1] (1.9) for h_n and C_n and then modified using (2.13) and (2.30).

Q&Q

For completeness let us mention that in [33] there is given a very complicated linearization formula for Al-Salam–Chihara polynomials given in Theorem 1.

3.2.2. *Useful finite sums.* We have also the following a very useful generalization of formula (1.12) of [5] which was proved in [36] (Lemma2 assertion i)).

$\forall n\geq 0:$

(3.20)

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q B_{n-k}(x|q)H_{k+m}(x|q)=\begin{cases}0 & if\quad n>m\\ (-1)^n q^{\binom{n}{2}}\frac{[m]_q!}{[m-n]_q!}H_{m-n}(x|q) & if\quad m\geq n\end{cases}$$

Let us remark that for $q=0$ (3.20) reduces to 3-term recurrence of polynomials $U_n(x/2)$.

For $q=1$ we get

$$\sum_{k=0}^{n}\binom{n}{k}i^{n-k}H_{n-k}(ix)H_{k+m}(x)=\begin{cases}0 & if\quad n>m\\ (-1)^n\frac{m!}{(m-n)!}H_{m-n}(x) & if\quad m\geq n\end{cases}.$$

4. Infinite expansions

4.1. **Kernels.** We start with the famous Poisson–Mehler expansion of $f_{CN}(x|y,\rho,q)/f_N(x|q)$ in the an infinite series of Mercier's type (compare e.g. [26]). Namely the following fact is true:

Theorem 1. $\forall |q|, |\rho| < 1; x, y \in S(q)$:

$$\frac{(\rho^2)_\infty}{\prod_{k=0}^{\infty} W_k(x, y, \rho|q)} = \sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!} H_n(x|q) H_n(y|q), \tag{4.1}$$

For $q = 1$, $x, y \in \mathbb{R}$ we have

$$\frac{\exp\left(\frac{x^2+y^2}{2}\right)}{\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 + y^2 - 2\rho xy}{2(1-\rho^2)}\right) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} H_n(x) H_n(y) \tag{4.2}$$

Proof. There exist many proofs of both formulae (see e.g. [17], [2]). One of the shortest, exploiting connection coefficients, given in (3.3) is given in [35]. □

Corollary 1. $\forall |q|, |\rho| < 1; x \in S(q) : \sum_{k\geq 0} \frac{\rho^k (\rho q^{k-1})_\infty}{[k]_q!} H_{2k}(x|q) = \frac{(\rho^2)_\infty}{(\rho)_\infty} \prod_{k=0}^{\infty} L_k^{-1}(x, \rho|q)$

Proof. We put $y = x$ in (4.1), then we apply (3.13), change order of summation and finally apply formulae $\frac{1}{(\rho)_{j+1}} = \sum_{k\geq 0} \begin{bmatrix} j+k \\ k \end{bmatrix}_q \rho^k$ and $\frac{(\rho)_\infty}{(\rho)_{j+1}} = (q^{j-1}\rho)_\infty$ □

We will call expression of the form of the right hand side of (4.1) the kernel expansion while the expressions from the left hand side of (4.1) kernels. The name refers to Mercier's theorem and the fact that for example $\int_{S(q)} k(x, y|\rho, q) H_n(x|q) f_N(x|q) dx = \rho^n H_n(y|q) f_N(y|q)$, where we denoted by $k(x, y|\rho, q)$ the left hand side of (4.1). Hence we see that k is a kernel, while function $H_n(x|q) f_N(x|q)$ are eigenfunctions of k with ρ^n being eigenvalue related to eigenfunction $H_n(x|q) f_N(x|q)$. Such kernels and kernel expansions are very important in quantum physics in particular in the analysis of different models of harmonic oscillator.

In the literature however there is small confusion concerning terminology. Sometimes expression of the form $\sum_{n\geq 0} a_n p_n(x) p_n(y)$ where $\{p_n\}$ is is a family of polynomials are also called kernels (like in [40])) or even sometimes 'bilinear generating function' (see e.g. [28])) or also Poisson kernel. If say $p_n(y)$ is substituted by say $q_n(y)$ then one says that we deal with the non-symmetric kernel.

The process of expressing these sums in a closed form is then called 'summing of kernels'.

Summing the kernel expansions is difficult. Proving positivity of the kernels is another difficult problem. Only some are known and have relatively simple forms. In most cases sums are in the form of a complex finite sum of the so called basic hypergeometric functions. Below we will present several of them. Mostly the ones involving the big q-Hermite, Al-Salam–Chihara and q-ultraspherical polynomials.

To present more complicated sums we will need the following definition of the basic hypergeometric function namely

$$ {}_j\phi_k\left[\begin{array}{cccc} a_1 & a_2 & \dots & a_j \\ b_1 & b_2 & \dots & b_k \end{array}; q, x\right] = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_j|q)}{(b_1, \dots, b_k|q)} \left((-1)^n q^{\binom{n}{2}}\right)^{1+k-j} x^n \tag{4.3}$$

(4.4)

$$ {}_{2m}W_{2m-1}(a, a_1, \dots, a_{2m-3}; q, x) = {}_{2m}\phi_{2m-1}\left[\begin{array}{ccccccc} a & q\sqrt{a} & -q\sqrt{a} & a_1 & a_2 & \dots & a_{2m-3} \\ \sqrt{a} & -\sqrt{a} & \frac{qa}{a_1} & \frac{qa}{a_2} & \dots & \frac{qa}{a_{2m-3}} & \end{array}; q, x\right]$$

We will now present the kernels built of families of polynomials that are discussed here and their sums.

Theorem 2. *i)* $\forall\left|t\right|<1,\left|x\right|,\left|y\right|<2:$

$$\sum_{n=0}^{\infty}t^{n}U_{n}\left(x/2\right)U_{n}\left(y/2\right)=\frac{\left(1-t^{2}\right)}{\left(\left(1-t^{2}\right)^{2}-t\left(1+t^{2}\right)xy+t^{2}(x^{2}+y^{2})\right)}.$$

ii) $\forall\left|t\right|<1,\left|x\right|,\left|y\right|<1:$

$$\sum_{n=0}^{\infty}\frac{\left(1-\beta q^{n}\right)\left(q\right)_{n}}{\left(1-\beta\right)\left(\beta^{2}\right)_{n}}t^{n}C_{n}\left(x|\beta,q\right)C_{n}\left(y|\beta,q\right)=$$
$$\frac{\left(\beta q\right)_{\infty}^{2}}{\left(\beta^{2}\right)_{\infty}\left(\beta t^{2}\right)_{\infty}}\prod_{n=0}^{\infty}\frac{w_{n}\left(x,y|t\beta,q\right)}{w_{n}\left(x,y|t,q\right)}$$
$$\times\ {}_{8}W_{7}\left(\frac{\beta t^{2}}{q},\frac{\beta}{q},te^{i(\theta+\phi)},te^{-i(\theta+\phi)},te^{i(\theta-\phi)},te^{-i(\theta-\phi)};q,\beta q\right),$$

where $x=\cos\theta,\ y=\cos\phi$.

iii) $\forall\ \left|x\right|,\left|y\right|,\left|t\right|,\left|tb/a\right|\leq1$ *:*

$$\sum_{n\geq0}\frac{\left(tb/a\right)^{n}}{\left(q\right)_{n}}h_{n}\left(x|a,q\right)h_{n}\left(y|b,q\right)=\left(\frac{b^{2}t^{2}}{a^{2}}\right)_{\infty}\prod_{k=0}^{\infty}\frac{v_{k}\left(x,tb|q\right)}{w_{k}\left(x,y,t\frac{b}{a}|q\right)}\times \tag{4.5}$$
$${}_{3}\phi_{2}\left(\begin{array}{ccc}t & bte^{i(\theta+\phi)}/a & bte^{i(-\theta+\phi)}/a\\ b^{2}t^{2}/a^{2} & bte^{i\phi} & \end{array};q,be^{-i\phi}\right),$$

with $x=\cos\theta$ *and* $y=\cos\phi$.

iv) $\forall\left|t\right|<1,x,y\in S\left(q\right),ab=\alpha\beta:$

$$\sum_{n\geq0}\frac{\left(t\alpha/a\right)^{n}}{\left(q\right)_{n}\left(ab\right)_{n}}Q_{n}\left(x|a,b,q\right)Q_{n}\left(y|\alpha,\beta,q\right)=$$
$$\frac{\left(\frac{\alpha^{2}t^{2}}{a},\frac{\alpha^{2}t}{a}e^{i\theta},be^{-i\theta},bte^{i\theta},\alpha te^{-i\phi},\alpha te^{i\phi}\right)_{\infty}}{\left(ab,\frac{\alpha^{2}t^{2}}{a}e^{i\theta}\right)_{\infty}\prod_{k=0}^{\infty}w_{k}\left(x,y|\frac{\alpha t}{a},q\right)}\ {}_{8}W_{7}\left(\frac{\alpha^{2}t^{2}e^{i\theta}}{aq},t,\frac{\alpha t}{\beta},ae^{i\theta},\frac{\alpha t}{a}e^{i(\theta+\phi)},\frac{\alpha t}{a}e^{i(\theta-\phi)};q,be^{-i\theta}\right)$$

where as before $x=\cos\theta$ *and* $y=\cos\phi$ *and*

$$\sum_{n\geq0}\frac{t^{n}}{\left(q\right)_{n}\left(ab\right)_{n}}Q_{n}\left(x|a,b,q\right)Q_{n}\left(y|\alpha,\beta,q\right)=$$
$$\frac{\left(\frac{\beta t}{a}\right)_{\infty}}{\left(\alpha at\right)_{\infty}}\prod_{k=0}^{\infty}\frac{(1+\alpha^{2}t^{2}q^{2k})^{2}-2\alpha tq^{k}\left(x+y\right)\left(1+\alpha^{2}t^{2}q^{2k}\right)+4\alpha^{2}xyt^{2}q^{2k}}{w_{k}\left(x,y|t,q\right)}$$
$${}_{8}W_{7}\left(\frac{\alpha at}{q},\frac{\alpha t}{b},ae^{i\theta},ae^{-i\theta},\alpha e^{i\phi},\alpha e^{-i\phi};q;\frac{\beta t}{a}\right).$$

v) For $|\rho_1|, |\rho_2|, |q| < 1,\ x, y \in S(q)$

(4.6)
$$0 \leq \sum_{n\geq 0} \frac{\rho_1^n}{[n]_q! (\rho_2^2)_n} P_n(x|y, \rho_2, q) P_n\left(z|y, \frac{\rho_2}{\rho_1}, q\right) = \frac{(\rho_1^2)_\infty}{(\rho_2^2)_\infty} \prod_{k=0}^{\infty} \frac{W_k(x, z|\rho_2, q)}{W_k(x, y|\rho_1, q)}.$$

Remarks concerning the proof. i) We set $q = 0$ in (4.1) and use (2.15). ii) It is formula (1.7) in [28] based on [16]. iii) it is formula (14.14) in [40], iv) these are formulae (14.5) and (14.8) of [40]. v) Notice that it cannot be derived from assertion iv) since the condition $ab = \alpha\beta$ is not satisfied. Recall that (see (2.22)) $ab = \rho_2^2$ while $\alpha\beta = \rho_1^2$. For the proof recall the idea of expansion of ratio of densities presented in [35], use formulae (3.7) and (2.28) and finally notice that $f_{CN}(x|y, \rho_1 q) / f_{CN}(x|z, \rho_2, q) = \frac{(\rho_1^2)_\infty}{(\rho_2^2)_\infty} \prod_{k=0}^{\infty} \frac{W_k(x,z|\rho_2,q)}{W_k(x,y|\rho_1,q)}$. □

Corollary 2. $\forall\ |a| > |b|,\ x, y \in S(q):$
$$0 \leq \sum_{n\geq 0} \frac{b^n}{[n]_q! a^n} H_n(x|a, q) H_n(y|b, q) = \left(\frac{b^2}{a^2}\right)_\infty \prod_{k=0}^{\infty} \frac{V_k(x, b|q)}{W_k\left(x, y|\frac{b}{a}, q\right)}.$$

Proof. we set $t = 0$ in (4.5) and assume $|b| < |a|$. For an alternative simple proof see [37]. □

4.2. **Other infinite expansions.** In this subsection we will present some expansions that can be viewed as reciprocals of some presented above expansions and also some generalizations of so called Kibble–Slepian formula.

We start with some reciprocals of the expansion obtained above.

4.2.1. *Expansions of kernel's reciprocals.*

Theorem 3. *i) For* $|q|, |\rho| < 1, x, y \in S(q):$
$$1/\sum_{n=0}^{\infty} \frac{\rho^n}{[n]_q!} H_n(x|q) H_n(y|q) = \sum_{n=0}^{\infty} \frac{\rho^n}{(\rho^2)_n [n]_q!} B_n(y|q) P_n(x|y, \rho, q).$$

ii) For $x, y \in \mathbb{R}$ *and* $\rho^2 < 1/2$
$$1/\sum_{n=0}^{\infty} \frac{\rho^n}{n!} H_n(x) H_n(y) = \sum_{n=0}^{\infty} \frac{\rho^n i^n}{n! (1-\rho^2)^{n/2}} H_n(ix) H_n\left(\frac{(x-\rho y)}{\sqrt{1-\rho^2}}\right).$$

iii) For $|q| < 1, |a| < |b|, x, y \in S(q):$
$$1/\sum_{n\geq 0} \frac{a^n}{[n]_q! b^n} H_n(x|a, q) H_n(y|b, q) = \sum_{n\geq 0} \frac{a^n}{[n]_q! b^n (a^2/b^2)_n} B_n(y|b, q) P_n(x|y, a/b, q)$$

iv) For $|\rho_1|, |\rho_2|, |q| < 1,\ x, y \in S(q):$
$$1/\sum_{n\geq 0} \frac{\rho_1^n}{[n]_q! (\rho_2^2)_n} P_n(x|y, \rho_2, q) P_n\left(z|y, \frac{\rho_2}{\rho_1}, q\right) = \sum_{n\geq 0} \frac{\rho_2^n}{[n]_q! (\rho_1^2)_n} P_n(x|z, \rho_1, q) P_n\left(y|z, \frac{\rho_1}{\rho_2}, q\right)$$

Remarks concerning the proof. i) and ii) are proved in [35]. iii) is proved in [37]. iv) directly follows (4.6) □

4.2.2. *Some auxiliary infinite expansions.* The result below can be viewed as summing certain non-symmetric kernel.

Lemma 1. *For* $x, y \in S(q)$, $|\rho| < 1$ *let us denote* $\gamma_{m,k}(x, y|\rho, q) = \sum_{k=0}^{\infty} \frac{\rho^k}{[k]_q!} H_{k+m}(x|q) H_{k+k}(y|q)$ *. Then*

$$\gamma_{m,k}(x, y|\rho, q) = \gamma_{0,0}(x, y|\rho, q) Q_{m,k}(x, y|\rho, q), \tag{4.7}$$

where $Q_{m,k}$ *is a polynomial in* x *and* y *of order at most* $m + k$.
Further denote $C_n(x, y|\rho_1, \rho_2, \rho_3, q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \rho_1^{n-k} \rho_2^k Q_{n-k,k}(x, y|\rho_{3,} q)$.
Then we have in particular

i) $Q_{m,k}(x, y|\rho, q) = Q_{k,m}(y, x|\rho, q)$ *and*

$$Q_{m,k}(x, y|\rho, q) = \sum_{s=0}^{k} (-1)^s q^{\binom{s}{2}} \begin{bmatrix} k \\ s \end{bmatrix}_q \rho^s H_{k-s}(y|q) P_{m+s}(x|y, \rho, q)/(\rho^2)_{m+s},$$

ii)

$$C_n(x, y|\rho_1, \rho_2, \rho_3, q) = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix}_q H_{n-s}(y|q) P_s(x|y, \rho_3, q) \rho_1^{n-s} \rho_2^s (\rho_1\rho_3/\rho_2)_s / \left(\rho_3^2\right)_s.$$

Proof. Proof that $\gamma_{m,k}(x, y|\rho, q) / \gamma_{0,0}(x, y|\rho, q)$ is a polynomial can be deduced from [9] (formula 1.4) where the result was formulated for Rogers-Szegö polynomials. To get the ii) from this result of Carlitz using (2.35) is not easy. For the alternative, simple although lengthy proof of the general case and other assertions we refer the reader to [38] and [36]. □

Remark 2. As mentioned above in [9] one can find formula 1.4 that reads:

$$\sum_{k\geq 0} \frac{t^k}{(q)_k} s_{n+k}(x|q) s_{m+k}(y|q) = \frac{\left(xyt^2\right)_\infty}{(t, xt, yt, xyt)_\infty} \sum_{k=0}^{m} \sum_{j=0}^{n} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(xt)_k (yt)_j (xyt)_{k+j}}{(xyt^2)_{k+j}} x^{m-k} y^{n-j}.$$

Using (2.35) we arrive at:

$$\begin{aligned} \sum_{k=0}^{\infty} h_{m+k}(x|q) h_{n+k}(y|q) \frac{t^k}{(q)_k} &= \prod_{n=0}^{\infty} \frac{\left(1 - t^2 q^n\right)}{w_n(x, y|t, q)} \\ \times \sum_{k=0}^{m} \sum_{l=0}^{n} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q & \frac{\left(te^{i(-\theta+\eta)}\right)_k \left(te^{i(\theta-\eta)}\right)_l \left(te^{-i(\theta+\eta)}\right)_{k+l}}{(t^2)_{k+l}} e^{-i(m-2k)\theta} e^{-i(n-2l)\eta} \end{aligned} \tag{4.8}$$

where as usually $x = \cos\theta$, $y = \cos\eta$. The difference between $\gamma_{m,k}$ from lemma 1 and the left hand side of (4.8) is that we use substitution $x- > 2x/\sqrt{1-q}$, $y- > 2y/\sqrt{1-q}$, $t- > t/(1-q)$. Hence comparing the right hand side of (4.8) and modified assertion i) of Lemma 1, we get

$$\begin{aligned} &\sum_{k=0}^{m} \sum_{l=0}^{n} \begin{bmatrix} m \\ k \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q \frac{\left(te^{i(-\theta+\eta)}\right)_k \left(te^{i(\theta-\eta)}\right)_l \left(te^{-i(\theta+\eta)}\right)_{k+l}}{(t^2)_{k+l}} e^{-i(m-2k)\theta} e^{-i(n-2l)\eta} \\ &\sum_{j=0}^{n} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q t^j h_{n-j}(y|q) p_{m+j}(x|y, t, q) / \left(t^2\right)_{j+m}, \end{aligned} \tag{4.9}$$

proving usefulness of the polynomials p_n also neither they do have direct probabilistic interpretation as polynomials P_n nor they have symmetry with respect to

the parameters y and t as the ordinary ASC polynomials do. Note also that the direct proof of (4.9) is not a simple thing.

4.2.3. *Generalization of Kibble–Slepian formula.* Recall that Kibble in 1949 [22] and independently Slepian in 1972 [32] extended the Poisson–Mehler formula to higher dimensions, expanding ratio of the standardized multidimensional Gaussian density divided by the product of one dimensional marginal densities in the multiple sum involving only constants (correlation coefficients) and the Hermite polynomials. The formula in its generality can be found in [17] (4.7.2 p.107). Since we are going to generalize its 3-dimensional version only this version will be presented here.

Namely let us consider 3 dimensional density $f_{3D}\left(x_1,x_2,x_3;\rho_{12},\rho_{13},\rho_{23}\right)$ of Normal random vector $N\left(\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1&\rho_{12}&\rho_{13}\\\rho_{12}&1&\rho_{23}\\\rho_{13}&\rho_{23}&1\end{bmatrix}\right)$. Of course we must assume that the parameters ρ_{12}, ρ_{13}, ρ_{23} are such that the variance covariance matrix is positive definite i.e. such that

$$1+2\rho_{12}\rho_{13}\rho_{23}-\rho_{12}^2-\rho_{13}^2-\rho_{23}^2>0. \tag{4.10}$$

Then Kibble–Slepian formula reads that

$$\exp\left(\frac{x_1^2+x_2^2+x_3^2}{2}\right)f_{3D}\left(x_1,x_2,x_3;\rho_{12},\rho_{13},\rho_{23}\right)$$
$$=\sum_{k,m,m=0}^{\infty}\frac{\rho_{12}^k\rho_{13}^m\rho_{23}^n}{k!m!n!}H_{k+m}\left(x_1\right)H_{k+n}\left(x_2\right)H_{m+n}\left(x_3\right).$$

Thus immediate generalization of this formula would be to substitute the Hermite polynomials by the $q-$Hermite ones and factorials by the $q-$factorials.

The question is if such sum is positive. It turn out that not in general i.e. not for all ρ_{12}, ρ_{13}, ρ_{23} satisfying (4.10). Nevertheless it is interesting to compute the sum

$$\sum_{k,m,m=0}^{\infty}\frac{\rho_{12}^k\rho_{13}^m\rho_{23}^n}{[k]_q!\,[m]_q!\,[n]_q!}H_{k+m}\left(x_1|q\right)H_{k+n}\left(x_2|q\right)H_{m+n}\left(x_3|q\right). \tag{4.11}$$

For simplicity let us denote this sum by $g\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)$.

In [38] the following result have been formulated and proved.

Theorem 4. *i)*

$$g\left(x_1,x_2,x_3|\rho_{12},\rho_{13},\rho_{23},q\right)=\frac{\left(\rho_{13}^2\right)_\infty}{\prod_{k=0}^{\infty}W_k\left(x_1,x_3,\rho_{13}|q\right)}\sum_{s\geq0}\frac{1}{[s]_q!}H_s\left(x_2|q\right)C_s\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q\right)$$

where

$$C_n\left(x_1,x_3|\rho_{12},\rho_{23},\rho_{13},q\right)=\frac{1}{\left(\rho_{13}^2\right)_n}\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^kq^{\binom{k}{2}}\begin{bmatrix}n\\2k\end{bmatrix}_q\begin{bmatrix}2k\\k\end{bmatrix}_q[k]_q!\rho_{12}^k\rho_{13}^k\rho_{23}^k\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}\right)_k\left(\frac{\rho_{13}\rho_{23}}{\rho_{12}}\right)_k$$
$$\sum_{j=0}^{n-2k}\begin{bmatrix}n-2k\\j\end{bmatrix}_q\rho_{23}^j\left(\frac{\rho_{12}\rho_{13}}{\rho_{23}}q^k\right)_k\rho_{12}^{n-j-2k}\left(\frac{\rho_{13}\rho_{23}}{\rho_{12}}q^k\right)_{n-2k-j}H_j\left(x_1|q\right)H_{n-2k-j}\left(x_3|q\right)$$

similarly for other pairs $(1,3)$ *and* $(2,3)$,

ii)

$$g\left(x_1, x_2, x_3|\rho_{12}, \rho_{13}, \rho_{23}, q\right) = \frac{\left(\rho_{13}^2, \rho_{23}^2\right)_\infty}{\prod_{k=0}^{\infty} W_k\left(x_1, x_3, \rho_{13}|q\right) W_k(x_3, x_2, \rho_{23}|q)} \times \sum_{s=0}^{\infty} \frac{\rho_{12}^s \left(\rho_{13}\rho_{23}/\rho_{12}\right)_s}{[s]_q! \left(\rho_{13}^2\right)_s \left(\rho_{23}^2\right)_s} P_s\left(x_1|x_3, \rho_{13}, q\right) P_s\left(x_2|x_3, \rho_{23}, q\right), \tag{4.12}$$

similarly for other pairs $(1,3)$ *and* $(2,3)$*.*

Unfortunately as shown in [38], one can find such ρ_{12}, ρ_{13}, ρ_{23} that function g with these parameters assumes negative values for some $x_j \in S(q)$, $j = 1, 2, 3$ hence consequently $g\left(x_1, x_2, x_3|\rho_{12}, \rho_{13}, \rho_{23}, q\right) \prod_{j=0}^{3} f_N\left(x_j|q\right)$ with these values of parameters is not a density of a probability distribution.

Remark 3. Notice that if $\rho_{12} = q^m \rho_{13}\rho_{23}$ then the sum in 4.12 is finite having only m summands.

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